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THE CURVED INTERFACIAL CRACK BETWEEN DISSIMILAR ISOTROPIC SOLIDS

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Abstract—The paper examines analytically the role of curvature on the stress distribution of a curved interfacial crack between dissimilar isotropic solids. The crack-tip fields under in-plane and antiplane shear loading are studied, respectively. Using an asymptotic expansion of the circular interface geometry, the asymptotic solutions of the stress and displacement fields in the vicinity of the curved crack tip derived from modified stress functions is obtained. The eigenfunctions associated with the eigenvalues λ for the curved crack consist of not only r^{λ} terms, but also $r^{\lambda+1}$, $r^{\lambda+2}$, ... terms. In some cases, the terms $r^{\lambda+1}(\ln r)$, $r^{\lambda+2}(\ln r)$, etc. may also exist. Two examples, frictionless contact near the circular crack-tip under in-plane loading and circular interfacial crack subject to antiplane shear loading, are derived in a closed-form asymptotic solution to elucidate the curvature effect. The case of fully open interfacial crack is also briefly described. Comparing the eigenfunction solutions of straight interfaces, the curvature effect enters the stress fields from the third-order term of the asymptotic solution for both cases. The condition for the existence of the $r^{1/2}(\ln r)$ term in the circular interfacial crack with frictionless contact is presented explicitly. Copyright © 1997 Elsevier Science Ltd.

INTRODUCTION

Precise determination of failure mechanism and failure criteria requires rigorous stress analysis in the vicinity of the flaws in the material or structure. For plate-like composite structures with delamination, Williams (1959) first addressed a problem of interfacial cracks between dissimilar isotropic materials along a straight interface. The stress exponents including strength of singularity and associated angular distributions were obtained from eigenfunction techniques. The analysis of the interfacial crack results in a rapid oscillation of stresses in a region near the crack tip. This problem was further studied by Erdogan (1963), England (1965), Erdogan (1965), Rice and Sih (1965) and Comninou (1977). Recent papers by Rice (1988), Gautesen and Dundurs (1988), Suo and Hutchinson (1990), indicated a renewed interest of the subject and gave further assessments of this topic. The complete stress fields for delaminated composite plates can be determined by the exact stress exponents and angular distributions embedded in a special numerical scheme such as the singular hybrid element (Wang and Yuan, 1983).

For delamination in curved composites between curved layers or for debonding between the fiber and the matrix in the microscale, the stress field near the curved crack received relatively less attention. England (1966) studied a circular interfacial crack between dissimilar isotropic materials. The oscillatory singularity which is identical to that in the straight crack was also found. A similar statement was also made by Perlman and Sih (1967). Aksentian (1967) used an asymptotic expansion method to investigate the dissimilar isotropic solids joining along a three-dimensional curved wedge. In the vicinity of the curved wedge the order of the singularity appears to be the same as that in the two-dimensional plane wedge problem. Dempsey and Sinclair (1979) studied two-dimensional straight wedge problems with various wedge boundary conditions and derived conditions for existence of logarithmic singularity in homogeneous materials. Ting (1985) studied the stress distribution at the apex of two-dimensional curved wedges using asymptotic solutions. He concluded that the singularities remain the same as in the straight wedge, whereas the effect of curvature only changes the form of the eigenfunctions. In addition, the curved wedge may induce logarithmic eigenfunction terms in the higher order asymptotic expansions.

In this paper, the curvature effect on the crack-tip stress distribution between dissimilar isotropic solids is quantitatively identified. Two examples are demonstrated: circular interfacial crack between dissimilar isotropic solids under in-plane and antiplane loading. In particular, a circular interfacial crack under in-plane loading with frictionless contact and the composite under antiplane shear are solved in a closed form up to higher order asymptotic expansions to identify the curvature effect.

CURVED INTERFACE CRACK WITH FRICTIONLESS CONTACT

Consider a semi-infinite planar crack along the interface between two dissimilarly linear-elastic isotropic solids. The polar coordinate (r, θ) is located at the crack tip. Following Williams' approach (1957), a separation of variables solution for the stress function is suggested by $F = r^{\lambda+2}f(\theta, \lambda) = r^{\lambda+2}\mathbf{p}(\theta, \lambda)\mathbf{a}$ in terms of series of eigenfunctions. The stresses and displacements are then expressed as (Williams, 1959)

$$\boldsymbol{\sigma} = r^{\lambda}\mathbf{S}(\theta, \lambda)\mathbf{a}, \quad \mathbf{u} = r^{\lambda+1}\mathbf{U}(\theta, \lambda)\mathbf{a} \tag{1}$$

where

$$\boldsymbol{\sigma} = [\sigma_r \quad \sigma_\theta \quad \tau_{r\theta}]^{\mathrm{T}} \quad \mathbf{u} = [u_r \quad u_\theta]^{\mathrm{T}}$$

$$\mathbf{S} = \begin{bmatrix} \Sigma_r \\ \Sigma_\theta \\ \Sigma_{r\theta} \end{bmatrix}_{3\times 4} = \begin{bmatrix} (\lambda+2)\mathbf{p}+\mathbf{p}'' \\ (\lambda+1)(\lambda+2)\mathbf{p} \\ -(\lambda+1)\mathbf{p}' \end{bmatrix} \quad \mathbf{U} = \begin{bmatrix} \mathbf{U}_r \\ \mathbf{U}_\theta \end{bmatrix}_{2\times 4} = \frac{1}{2\mu}\begin{bmatrix} -(\lambda+2)\mathbf{p}+(\kappa+1)\mathbf{q}'/\lambda \\ -\mathbf{p}'+(\kappa+1)\mathbf{q} \end{bmatrix} \tag{2}$$

$$\mathbf{p} = [\sin(\lambda+2)\theta, \cos(\lambda+2)\theta, \sin\lambda\theta, \cos\lambda\theta] \tag{3}$$

$$\mathbf{q} = [0, 0, -\cos\lambda\theta, \sin\lambda\theta] \tag{4}$$

$$\mathbf{a} = [a_1, a_2, a_3, a_4]^{\mathrm{T}}$$

and μ is the shear modulus ; $\kappa = \begin{cases} \dfrac{3-\nu}{1+\nu} & \text{for plane stress} \\ 3-4\nu & \text{for plane strain} \end{cases}$.

The prime in eqns (2), and in the sequel, represents the derivative with respect to θ. The unknown vector $\mathbf{a}$ is dependent on λ. When the interfacial crack between dissimilar isotropic solids is lying on a straight line, the boundary conditions for the crack with frictionless contact shown in Fig. 1 are

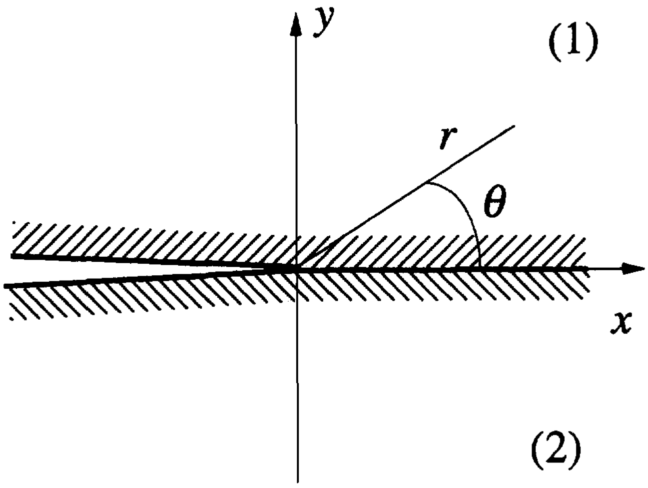

Fig. 1. Interfacial crack with straight boundaries.

$$\tau_{r\theta}^{(1)}(r,\pi) = 0, \quad \tau_{r\theta}^{(2)}(r,-\pi) = 0, \quad \sigma_{\theta}^{(1)}(r,\pi) = \sigma_{\theta}^{(2)}(r,-\pi), \quad u_{\theta}^{(1)}(r,\pi) = u_{\theta}^{(2)}(r,-\pi) \tag{5}$$

$$\tau_{r\theta}^{(1)}(r,0) = \tau_{r\theta}^{(2)}(r,0), \quad \sigma_{\theta}^{(1)}(r,0) = \sigma_{\theta}^{(2)}(r,0), \quad u_{r}^{(1)}(r,0) = u_{r}^{(2)}(r,0), \quad u_{\theta}^{(1)}(r,0) = u_{\theta}^{(2)}(r,0) \tag{6}$$

where all the quantities referred to material 1 are identified with superscript (1) and those referred to material 2 with superscript (2).

Using matrix notations, eqns (5) and (6) can be written as

$$\mathbf{K}_o(\lambda)\mathbf{A} = 0 \tag{7}$$

where the 8×8 matrix $\mathbf{K}_o(\lambda)$ is given in the Appendix and

$$\mathbf{A} = [\mathbf{a}^{(1)} \quad \mathbf{a}^{(2)}]^{\mathrm{T}}. \tag{8}$$

A nontrivial solution of eqn (7) leads to

$$\|\mathbf{K}_o(\lambda)\| = 0.$$

Evaluating the determinant, the admissible eigenvalues are given by Comninou (1977), that is

$$\sin^3 \lambda\pi \cos \lambda\pi = 0 \quad (\lambda > -1) \tag{9}$$

or

$$\lambda = n - \tfrac{1}{2}, \quad \lambda = n, \quad n = 0, 1, 2, 3, \ldots. \tag{10}$$

The eigenvectors and stress fields associated with the first three eigenvalues, $\lambda = -1/2$, 0, and 1/2 have also been obtained. In fact, the eigenvectors and stress fields corresponding to any eigenvalue can be conveniently expressed as

(a) eigenvalues $\lambda = n - 1/2$, $(n = 0, 1, 2, 3, \ldots)$

$$\mathbf{A} = \frac{k}{(\lambda+1)(\lambda+2)} \times [2+\lambda(1-\beta) \quad 0 \quad -(\lambda+2)(1-\beta) \quad 0 \quad 2+\lambda(1+\beta) \quad 0 \quad -(\lambda+2)(1+\beta) \quad 0]^{\mathrm{T}} \tag{11}$$

$$\begin{aligned}
\sigma_r^{(1)} &= -kr^{\lambda}\{[2+\lambda(1-\beta)]\sin(\lambda+2)\theta - (\lambda-2)(1-\beta)\sin\lambda\theta\} \\
\sigma_\theta^{(1)} &= kr^{\lambda}\{[2+\lambda(1-\beta)]\sin(\lambda+2)\theta - (\lambda+2)(1-\beta)\sin\lambda\theta\} \\
\tau_{r\theta}^{(1)} &= -kr^{\lambda}\{[2+\lambda(1-\beta)]\cos(\lambda+2)\theta - \lambda(1-\beta)\cos\lambda\theta\}
\end{aligned} \tag{12}$$

$$\begin{aligned}
\sigma_r^{(2)} &= -kr^{\lambda}\{[2+\lambda(1+\beta)]\sin(\lambda+2)\theta - (\lambda-2)(1+\beta)\sin\lambda\theta\} \\
\sigma_\theta^{(2)} &= kr^{\lambda}\{[2+\lambda(1+\beta)]\sin(\lambda+2)\theta - (\lambda+2)(1+\beta)\sin\lambda\theta\} \\
\tau_{r\theta}^{(2)} &= -kr^{\lambda}\{[2+\lambda(1+\beta)]\cos(\lambda+2)\theta - \lambda(1+\beta)\cos\lambda\theta\}
\end{aligned} \tag{13}$$

where k is an undetermined constant for each fractional eigenvalue;

(b) eigenvalues $\lambda = n$ (triple roots), $(n = 0, 1, 2, 3, \ldots)$

$$\mathbf{A} = c_1\mathbf{D}_1 + c_2\mathbf{D}_2 + c_3\mathbf{D}_3 \tag{14}$$

$$\begin{aligned}
\mathbf{D}_1 &\equiv [0 \quad 1+\alpha-(\lambda+2)(\alpha-\beta) \quad 0 \quad (\lambda+2)(\alpha-\beta) \quad 0 \quad 1+\alpha \quad 0 \quad 0]^{\mathrm{T}} \\
\mathbf{D}_2 &\equiv [0 \quad 1-\alpha \quad 0 \quad 0 \quad 0 \quad 1-\alpha+(\lambda+2)(\alpha-\beta) \quad 0 \quad -(\lambda+2)(\alpha-\beta)]^{\mathrm{T}} \\
\mathbf{D}_3 &\equiv [\lambda(1-\alpha) \quad 0 \quad -(\lambda+2)(1-\alpha) \quad 0 \quad \lambda(1+\alpha) \quad 0 \quad -(\lambda+2)(1+\alpha) \quad 0]^{\mathrm{T}}
\end{aligned} \tag{15}$$

$$\begin{aligned}
\sigma_r^{(1)} &= -c_1 r^{\lambda}\{[1+\alpha-(\lambda+2)(\alpha-\beta)]\cos(\lambda+2)\theta+(\lambda-2)(\alpha-\beta)\cos\lambda\theta\} \\
&\quad -c_2 r^{\lambda}(1-\alpha)\cos(\lambda+2)\theta \\
&\quad -c_3 r^{\lambda}(1-\alpha)[\lambda\sin(\lambda+2)\theta-(\lambda-2)\sin\lambda\theta] \\
\sigma_\theta^{(1)} &= c_1 r^{\lambda}\{[1+\alpha-(\lambda+2)(\alpha-\beta)]\cos(\lambda+2)\theta+(\lambda+2)(\alpha-\beta)\cos\lambda\theta\} \\
&\quad +c_2 r^{\lambda}(1-\alpha)\cos(\lambda+2)\theta \\
&\quad +c_3 r^{\lambda}(1-\alpha)[\lambda\sin(\lambda+2)\theta-(\lambda+2)\sin\lambda\theta] \\
\tau_{r\theta}^{(1)} &= c_1 r^{\lambda}\{[1+\alpha-(\lambda+2)(\alpha-\beta)]\sin(\lambda+2)\theta+\lambda(\alpha-\beta)\sin\lambda\theta\} \\
&\quad +c_2 r^{\lambda}(1-\alpha)\sin(\lambda+2)\theta \\
&\quad +c_3 r^{\lambda}(1-\alpha)\lambda[-\cos(\lambda+2)\theta+\cos\lambda\theta]
\end{aligned} \tag{16}$$

$$\begin{aligned}
\sigma_r^{(2)} &= -c_1 r^{\lambda}(1+\alpha)\cos(\lambda+2)\theta \\
&\quad -c_2 r^{\lambda}\{[1-\alpha+(\lambda+2)(\alpha-\beta)]\cos(\lambda+2)\theta-(\lambda-2)(\alpha-\beta)\cos\lambda\theta\} \\
&\quad -c_3 r^{\lambda}(1+\alpha)[\lambda\sin(\lambda+2)\theta-(\lambda-2)(1+\alpha)\sin\lambda\theta] \\
\sigma_\theta^{(2)} &= c_1 r^{\lambda}(1+\alpha)\cos(\lambda+2)\theta \\
&\quad +c_2 r^{\lambda}\{[1-\alpha+(\lambda+2)(\alpha-\beta)]\cos(\lambda+2)\theta-(\lambda+2)(\alpha-\beta)\cos\lambda\theta\} \\
&\quad +c_3 r^{\lambda}(1+\alpha)[\lambda\sin(\lambda+2)\theta-(\lambda+2)(1+\alpha)\sin\lambda\theta] \\
\tau_{r\theta}^{(2)} &= c_1 r^{\lambda}(1+\alpha)\sin(\lambda+2)\theta \\
&\quad +c_2 r^{\lambda}\{[1-\alpha+(\lambda+2)(\alpha-\beta)]\sin(\lambda+2)\theta-\lambda(\alpha-\beta)\sin\lambda\theta\} \\
&\quad +c_3 r^{\lambda}(1+\alpha)\lambda[-\cos(\lambda+2)\theta+\cos\lambda\theta]
\end{aligned} \tag{17}$$

where c_1, c_2, and c_3 are undetermined constants for each integer eigenvalue, respectively. α and β are the Dundurs' parameters defined as

$$\alpha = \frac{\mu_2(\kappa_1+1)-\mu_1(\kappa_2+1)}{\mu_2(\kappa_1+1)+\mu_1(\kappa_2+1)} \quad \beta = \frac{\mu_2(\kappa_1-1)-\mu_1(\kappa_2-1)}{\mu_2(\kappa_1+1)+\mu_1(\kappa_2+1)}.$$

The three-term asymptotic expansion of the stress field for the straight crack is written as

$$\boldsymbol{\sigma}^{(j)} = k_1 r^{-1/2}\mathbf{S}(\theta,-1/2)\mathbf{a}_1^{(j)}+\mathbf{S}(\theta,0)\mathbf{a}_2^{(j)}+k_3 r^{1/2}\mathbf{S}(\theta,1/2)\mathbf{a}_3^{(j)}+\cdots \quad j=1,2 \tag{18}$$

where $\mathbf{a}_1^{(j)}$, $\mathbf{a}_2^{(j)}$, and $\mathbf{a}_3^{(j)}$ are vectors which can be obtained from eqn (7) and the corresponding eigenvectors, $\mathbf{A}|_{\lambda=-1/2}$, $\mathbf{A}|_{\lambda=0}$, and $\mathbf{A}|_{\lambda=1/2}$ given in eqns (11) and (15). k_1, k_2 are arbitrary constants. Note that $\mathbf{a}_2^{(j)}$ contains two arbitrary constants c_1 and c_2 (see eqns (16) and (17)). The terms associated with c_3 are zero for $\lambda = 0$.

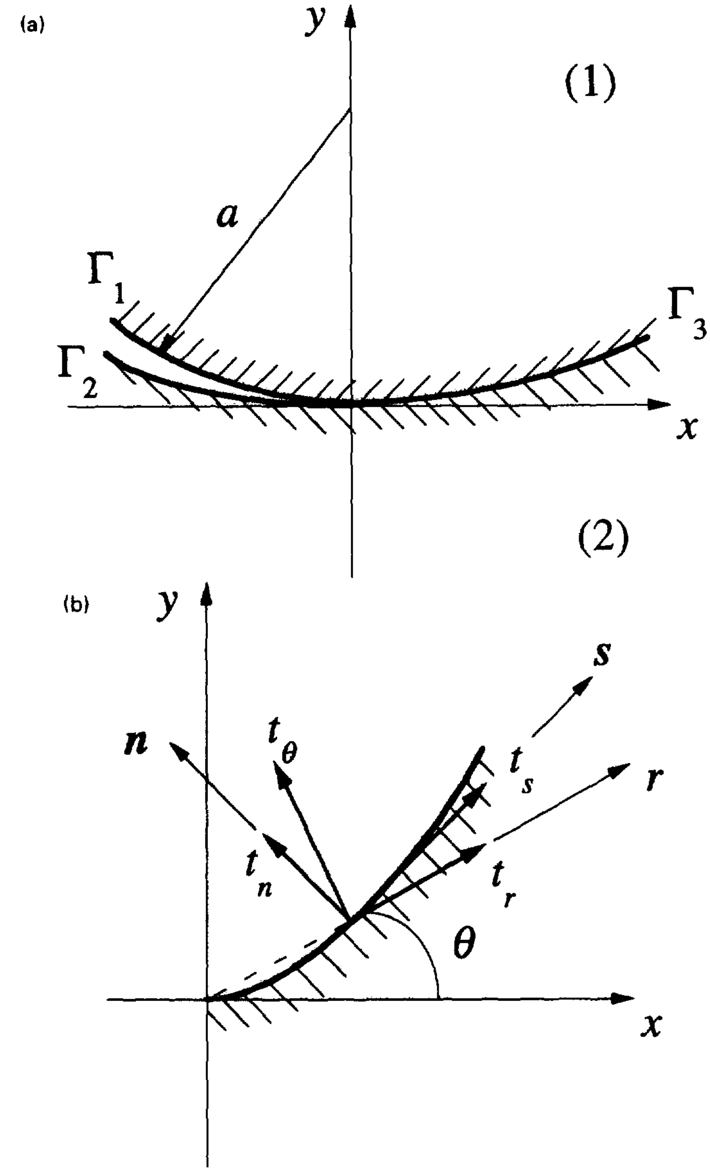

Fig. 2. (a) Interfacial crack lying on circular arc; (b) positive directions of **n**, (t_r, t_s), and (t_n, t_s).

The interface crack is lying on a smooth curved line shown in Fig. 2. The boundary conditions are

$$\begin{aligned} &t_s^{(1)} = 0 \quad \text{on } \Gamma_1 \qquad t_n^{(1)} = t_n^{(2)} \quad \text{on } \Gamma_1 \quad \text{and} \quad \Gamma_2 \\ &t_s^{(2)} = 0 \quad \text{on } \Gamma_2 \qquad u_n^{(1)} = u_n^{(2)} \quad \text{on } \Gamma_1 \quad \text{and} \quad \Gamma_2 \end{aligned} \tag{19}$$

$$\left.\begin{aligned} t_r^{(1)} &= -t_r^{(2)}, \quad t_\theta^{(1)} = -t_\theta^{(2)} \\ u_r^{(1)} &= u_r^{(2)}, \quad u_\theta^{(1)} = u_\theta^{(2)} \end{aligned}\right\} \text{on } \Gamma_3 \tag{20}$$

where (t_θ, t_r) are the components of surface tractions **t**, (t_s, t_r) are the shear and normal tractions on the crack surfaces, and

$$t_\theta = \sigma_\theta n_\theta + \tau_{r\theta} n_r, \quad t_r = \tau_{r\theta} n_\theta + \sigma_r n_r \tag{21}$$

$$t_s = t_r n_\theta - t_\theta n_r, \quad t_n = t_r n_r + t_\theta n_\theta \tag{22}$$

(n_r, n_θ) are the components of the unit outward normal $\mathbf{n}$ to the boundaries. Assume that in the vicinity of the crack tip the angle θ on each curved boundary Γ_i can be asymptotically expressed by

$$\theta = \theta_o + \theta_1 r + \theta_2 r^2 + \theta_3 r^3 + \cdots \quad \text{as } r \to 0 \text{ on } \Gamma_i. \tag{23}$$

For an interface crack lying on a circular arc with the radius of curvature being a, the equation of a circle of radius a centered at $(0, a)$ is

$$r = 2a \sin\theta, \quad 0 < \theta < 2\pi.$$

The values of θ_o, θ_1, θ_2, ... in eqn (23) are given by

$$\theta_o = \pi, \quad \theta_1 = \frac{-1}{2a}, \quad \theta_2 = 0, \quad \theta_3 = \frac{-1}{48a^3}, \ldots \quad \text{for } \Gamma_1$$

$$\theta_o = -\pi, \quad \theta_1 = \frac{-1}{2a}, \quad \theta_2 = 0, \quad \theta_3 = \frac{-1}{48a^3}, \ldots \quad \text{for } \Gamma_2$$

$$\theta_o = 0, \quad \theta_1 = \frac{1}{2a}, \quad \theta_2 = 0, \quad \theta_3 = \frac{1}{48a^3}, \ldots \quad \text{for } \Gamma_3. \tag{24}$$

It follows from (23) and $n_r = r(r^2 + r'^2)^{-1/2}$, $n_\theta = -r'(r^2 + r'^2)^{-1/2}$ that

$$n_r/n_\theta = -r\frac{\mathrm{d}\theta}{\mathrm{d}r} = -\theta_1 r - 2\theta_2 r^2 - \cdots. \tag{25}$$

On the curved boundaries, the assumed stresses and displacements can still be represented in the form (1). However, the values of θ do not remain fixed. Using Taylor's theorem, $\mathbf{S}$ and $\mathbf{U}$ on the boundaries can be expanded as

$$\mathbf{S}(\theta, \lambda) = \mathbf{S}(\theta_o, \lambda) + \theta_1 \mathbf{S}'(\theta_o, \lambda) r + [2\theta_2 \mathbf{S}'(\theta_o, \lambda) + \theta_1^2 \mathbf{S}''(\theta_o, \lambda)] r^2/2 + \cdots$$

$$\mathbf{U}(\theta, \lambda) = \mathbf{U}(\theta_o, \lambda) + \theta_1 \mathbf{U}'(\theta_o, \lambda) r + [2\theta_2 \mathbf{U}'(\theta_o, \lambda) + \theta_1^2 \mathbf{U}''(\theta_o, \lambda)] r^2/2 + \cdots. \tag{26}$$

In order to satisfy the boundary conditions on the curved lines Γ_i, consider the following form of stress function for a given λ (Ting, 1985, Dempsey and Sinclair, 1979)

$$F = r^{\lambda+2}\mathbf{p}(\theta, \lambda)\mathbf{a} + \frac{\partial}{\partial\lambda}[r^{\lambda+3}\mathbf{p}(\theta, \lambda+1)\mathbf{b}] + \frac{\partial}{\partial\lambda}[r^{\lambda+4}\mathbf{p}(\theta, \lambda+2)\mathbf{c}] + \cdots \tag{27}$$

where $\mathbf{b}$ and $\mathbf{c}$, ... are assumed to depend on λ. The stress function yields the expressions for stress and displacements

$$\boldsymbol{\sigma} = r^{\lambda}\mathbf{S}(\theta, \lambda)\mathbf{a} + r^{\lambda+1}\left\{(\ln r)\mathbf{S}(\theta, \lambda+1)\mathbf{b} + \frac{\partial}{\partial\lambda}[\mathbf{S}(\theta, \lambda+1)]\mathbf{b} + \mathbf{S}(\theta, \lambda+1)\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}\lambda}\right\} + O(r^{\lambda+2}) \tag{28}$$

$$\mathbf{u} = r^{\lambda+1}\mathbf{U}(\theta, \lambda)\mathbf{a} + r^{\lambda+2}\left\{(\ln r)\mathbf{U}(\theta, \lambda+1)\mathbf{b} + \frac{\partial}{\partial\lambda}[\mathbf{U}(\theta, \lambda+1)]\mathbf{b} + \mathbf{U}(\theta, \lambda+1)\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}\lambda}\right\} + O(r^{\lambda+3}). \tag{29}$$

Substituting (28) and (29) into (19) and (20), using (24)–(25), and equating coefficients of like powers of r in each equation, we have 24 linear equations for **a**, **b**, and $\mathrm{d}\mathbf{b}/\mathrm{d}\lambda$ which can be written in the matrix form:

$$\mathbf{K}_o(\lambda)\mathbf{A} = 0 \tag{30}$$

$$\mathbf{K}_o(\lambda+1)\mathbf{B} = 0 \tag{31}$$

$$\mathbf{K}_o(\lambda+1)\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}\lambda} = \frac{1}{2a}\mathbf{K}_1(\lambda)\mathbf{A} - \left[\frac{\partial}{\partial\lambda}\mathbf{K}_o(\lambda+1)\right]\mathbf{B} \tag{32}$$

where

$$\mathbf{B} = [\mathbf{b}^{(1)} \quad \mathbf{b}^{(2)}]^{\mathrm{T}} \quad \mathbf{b}^{(\alpha)} = [b_1^{(\alpha)}, b_2^{(\alpha)}, b_3^{(\alpha)}, b_4^{(\alpha)}]^{\mathrm{T}}. \tag{33}$$

with $\mathbf{K}_o(\lambda)$ and $\mathbf{K}_1(\lambda)$ given in the Appendix.

Since eqn (30) is identical to eqn (7), λ and **A** remain the same as the case of interfacial crack with straight boundaries. Equations (31) and (32) govern **B** and $\mathrm{d}\mathbf{B}/\mathrm{d}\lambda$ which represent the curvature effect. The eigenvalues determined by eqn (30) form an infinite sequence λ_1, $\lambda_2, \ldots$ and can be ordered $-1 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$ accordingly.

First, consider the solutions for **B** and $\mathrm{d}\mathbf{B}/\mathrm{d}\lambda$ associated with the stress singularity $\lambda = -1/2$. Since

$$\|\mathbf{K}_o(\lambda+1)\| = 0$$

$$\frac{\partial}{\partial\lambda}\|\mathbf{K}_o(\lambda+1)\| \neq 0$$

and the rank of $\mathbf{K}_o(\lambda+1)$ is 7, **B** has a unique solution (Zwiers *et al.*, 1982, Dempsey and Sinclair, 1979). Because $\lambda = -1/2$ and $\lambda+1 = 1/2$ are both eigenvalues, from eqns (30), (31), **A** and **B** can be symbolically rewritten as

$$\mathbf{A} = k\mathbf{A}_\lambda, \quad \mathbf{B} = e\frac{k}{2a}\mathbf{A}_{\lambda+1} \tag{34}$$

where $\mathbf{A}_\lambda$ and $\mathbf{A}_{\lambda+1}$ are eigenvectors corresponding to λ and $\lambda+1$, respectively, and are given in (11), k and e are proportional constants.

Inserting eqn (34) into eqn (32), the solution for $\mathrm{d}\mathbf{B}/\mathrm{d}\lambda$ can be written in the form

$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}\lambda} = \frac{k}{2a}\left(\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}\lambda}\right)_p + c\mathbf{A}_{1/2} \tag{35}$$

where c is an arbitrary constant and $(\mathrm{d}\mathbf{B}/\mathrm{d}\lambda)_p$ is a particular solution for the following equation

$$\mathbf{K}_o(\lambda+1)\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}\lambda} = \mathbf{K}_1(\lambda)\mathbf{A}_\lambda - \frac{\partial}{\partial\lambda}[\mathbf{K}_o(\lambda+1)]e\mathbf{A}_{\lambda+1} \tag{36}$$

where $\mathbf{K}_1(\lambda)\mathbf{A}_\lambda$ and

$$\left[\frac{\partial}{\partial\lambda}\mathbf{K}_o(\lambda+1)\right]\mathbf{A}_{\lambda+1}$$

for $\lambda = -1/2$ are given in the Appendix.

The value of e can be uniquely determined from the solvability condition of eqn (36). That is, the ranks of $\mathbf{K}_o(\lambda+1)$ and the augmented matrix

$$\left[\mathbf{K}_o(\lambda+1) \vdots \mathbf{K}_1(\lambda)\mathbf{A}_\lambda - \frac{\partial}{\partial\lambda}[\mathbf{K}_o(\lambda+1)]e\mathbf{A}_{\lambda+1}\right]$$

are the same (Hilderbrand, 1965). After extensively mathematical manipulation, the solvability condition of eqn (36) leads to

$$e = \frac{4}{\pi}\beta \tag{37}$$

while the solution of $(\mathrm{d}\mathbf{B}/\mathrm{d}\lambda)_p$ is

$$\left(\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}\lambda}\right)_p = \left[\left(\frac{\mathrm{d}\mathbf{b}^{(1)}}{\mathrm{d}\lambda}\right)_p \left(\frac{\mathrm{d}\mathbf{b}^{(2)}}{\mathrm{d}\lambda}\right)_p\right]^{\mathrm{T}}$$

where

$$\left(\frac{\mathrm{d}\mathbf{b}^{(1)}}{\mathrm{d}\lambda}\right)_p = \begin{bmatrix} -\frac{64}{75}\frac{\beta^2}{\pi} \\ -(1+3\beta) \\ 0 \\ (1+9\beta) \end{bmatrix}, \quad \left(\frac{\mathrm{d}\mathbf{b}^{(2)}}{\mathrm{d}\lambda}\right)_p = \begin{bmatrix} \frac{64}{75}\frac{\beta^2}{\pi} \\ -(1-3\beta) \\ 0 \\ (1-9\beta) \end{bmatrix}. \tag{38}$$

Similarly, we can determine the other higher-order terms with order $O(r^{\lambda+2}), \ldots$ in eqn (28) for $\lambda = -1/2$. Repeating the same procedure for $\lambda = \lambda_2, \lambda_3, \ldots$ and substituting the solutions for λ, $\mathbf{A}$, $\mathbf{B}$, and $\mathrm{d}\mathbf{B}/\mathrm{d}\lambda$ into eqn (28) leads to the asymptotic solutions for stresses in the case of curved cracks. The four-term expansion of the stress field for the circular interfacial crack with closed tip has the form

$$\begin{aligned}\boldsymbol{\sigma}^{(j)} &= k_1 r^{-1/2}\mathbf{S}(\theta, -1/2)\mathbf{a}_1^{(j)} + \mathbf{S}(\theta, 0)\mathbf{a}_2^{(j)} \\ &\quad + \frac{k_1}{2a} r^{1/2}\left\{e(\ln r)\mathbf{S}(\theta, 1/2)\mathbf{a}_3^{(j)} + e\frac{\partial}{\partial\lambda}[\mathbf{S}(\theta, \lambda+1)]|_{\lambda=-1/2}\mathbf{a}_3^{(j)} + \mathbf{S}(\theta, 1/2)\left(\frac{\mathrm{d}\mathbf{b}^{(j)}}{\mathrm{d}\lambda}\right)_p\Big|_{\lambda=-1/2}\right\} \\ &\quad + k_3 r^{1/2}\mathbf{S}(\theta, 1/2)\mathbf{a}_3^{(j)} + \cdots\end{aligned} \tag{39}$$

where k_1, k_3 are amplitudes.

Note that the undetermined constants k_i are functions of the curvature. The term associated with c in eqn (35) has been absorbed in the last term of eqn (39). It can be seen that the terms $r^{-1/2}$ and r° remain the first and the second term. The angular distributions for the first two terms are the same as the case of straight cracks. However, the third and higher order terms of circular cracks are different from that of straight cracks. Indeed, eqn (39) shows that the new $r^{1/2}(\ln r)$ term becomes the third order term; while the $r^{1/2}$ term which is the third term for the straight crack is the fourth order term for the circular crack.

In the case of $\beta = 0$, thereby $\mathbf{B} = 0$, eqn (39) reduces to

$$\boldsymbol{\sigma}^{(j)} = k_1 r^{-1/2}\mathbf{S}(\theta, -1/2)\mathbf{a}_1^{(j)} + \mathbf{S}(\theta, 0)\mathbf{a}_2^{(j)} + k_3 r^{1/2}\mathbf{S}(\theta, 1/2)\left[\mathbf{a}_3^{(j)} + \frac{k_1}{2ak_3}\left(\frac{\mathrm{d}\mathbf{b}^{(j)}}{\mathrm{d}\lambda}\right)_p\Big|_{\lambda=-1/2}\right] + \cdots \tag{40}$$

where

$$\left(\frac{\mathrm{d}\mathbf{b}^{(j)}}{\mathrm{d}\lambda}\right)_p\bigg|_{\lambda=-1/2} = [0 \quad -1 \quad 0 \quad 1]^{\mathrm{T}}.$$

Here the term $r^{1/2}$ becomes the third order term. However, the angular distribution of the third term for the circular crack is different from the crack with straight boundaries.

Let

$$\tilde{\sigma}_{ij3} = \Sigma_{ij}(\theta, 1/2)\left[\mathbf{a}_3 + \frac{k_1}{2ak_3}\left(\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}\lambda}\right)_p\bigg|_{\lambda=-1/2}\right]$$

then $\tilde{\sigma}_{ij3}$ calculated from eqns (12–13) represents explicitly the stress distribution of the third term for $\beta = 0$. Note that the superscript (j) has been dropped because the angular distribution in the case is independent of material properties. Figure 3 shows the stress angular distribution in terms of a polar diagram for different degrees of curvature effect $\xi = k_1/(2ak_3)$. A circle representing zero value is demonstrated in each figure for reference. The stress angular distribution for the straight crack ($a \to \infty$ or $\xi = 0$) is also shown for comparison. It can be seen that the distribution differs drastically as the curvature effect increases. The symmetric pattern of $\tilde{\sigma}_{r3}$ and $\tilde{\sigma}_{\theta 3}$ and antisymmetric angular distribution of $\tilde{\sigma}_{r\theta 3}$ for the straight crack has been distorted by the curvature.

The analysis can also be extended to a fully open interfacial crack. Allowing the eigenvalue λ to be complex-valued (Williams, 1959; Rice and Sih, 1965; and Symington, 1987), the new form of stress functions for the curved crack can be introduced as

$$F = \mathrm{Re}\left\{r^{\lambda+2}\mathbf{p}(\theta,\lambda)\mathbf{a} + \frac{\partial}{\partial\lambda}[r^{\lambda+3}\mathbf{p}(\theta,\lambda+1)\mathbf{b}] + \frac{\partial}{\partial\lambda}[r^{\lambda+4}\mathbf{p}(\theta,\lambda+2)\mathbf{c}] + \cdots\right\}. \tag{41}$$

The same matrix equations of eqns (28–32) in the fully open crack case can be obtained by performing a similar analysis. In this case, more compact forms of matrices $\mathbf{K}_o(\lambda)$ and $\mathbf{K}_1(\lambda)$ can be expressed by

$$\mathbf{K}_o(\lambda) = \begin{bmatrix} \mathbf{N}_o\mathbf{S}(\pi,\lambda) & 0 \\ 0 & \mathbf{N}_o\mathbf{S}(-\pi,\lambda) \\ \mathbf{N}_o\mathbf{S}(0,\lambda) & -\mathbf{N}_o\mathbf{S}(0,\lambda) \\ \mathbf{U}^{(1)}(0,\lambda) & -\mathbf{U}^{(2)}(0,\lambda) \end{bmatrix}$$

$$\mathbf{N}_o = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{42}$$

$$\mathbf{K}_1(\lambda) = \begin{bmatrix} \mathbf{N}_1\mathbf{S}(\pi,\lambda) - \mathbf{N}_o\mathbf{S}'(\pi,\lambda) & 0 \\ 0 & \mathbf{N}_1\mathbf{S}(-\pi,\lambda) - \mathbf{N}_o\mathbf{S}'(-\pi,\lambda) \\ -[\mathbf{N}_1\mathbf{S}(0,\lambda) - \mathbf{N}_o\mathbf{S}'(0,\lambda)] & \mathbf{N}_1\mathbf{S}(0,\lambda) - \mathbf{N}_o\mathbf{S}'(0,\lambda) \\ \mathbf{U}'^{(1)}(0,\lambda) & -\mathbf{U}'^{(2)}(0,\lambda) \end{bmatrix}$$

$$\mathbf{N}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \tag{43}$$

A nontrivial solution of $\mathbf{K}_o(\lambda) = 0$ leads to the characteristic equation (Symington, 1987)

$$(\lambda+2)\sin^2\lambda\pi[1 + (\beta^2 - 1)\sin^2\lambda\pi] = 0. \tag{44}$$

The admissible eigenvalues are given by

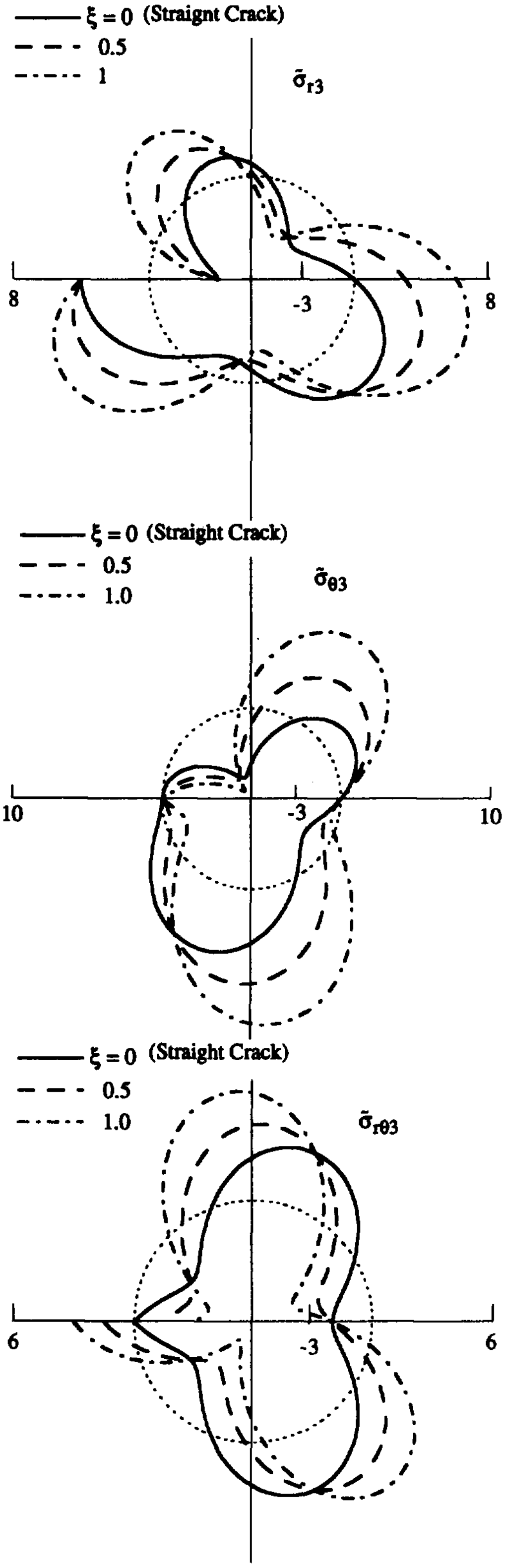

Fig. 3. Stress angular distribution of the third term asymptotic expansion for circular interfacial crack with frictionless contact under inplane loading.

$$\lambda = n-\tfrac{1}{2}\pm i\varepsilon, \quad \lambda = n, \quad n = 0,1,2,\ldots$$

$$\varepsilon = \frac{1}{2\pi}\ln\left(\frac{1+\beta}{1-\beta}\right) \tag{45}$$

where β is a Dundar's parameter.

Each eigenvalue in eqn (45) defines an eigenfunction. The eigenvectors $\mathbf{A}_\lambda = [\mathbf{a}^{(1)} \quad \mathbf{a}^{(2)}]^T$ associated with complex eigenvalues $\lambda = n-\frac{1}{2}\pm i\varepsilon$ can be expressed as

$$\mathbf{a}^{(1)} = \left[i\frac{1+(\lambda+1)e^{-2\pi\varepsilon}}{\lambda+2}, \quad \frac{1-(\lambda+1)e^{-2\pi\varepsilon}}{\lambda+2}, \quad -ie^{-2\pi\varepsilon}, \quad e^{-2\pi\varepsilon}\right]^T \eta$$

$$\mathbf{a}^{(2)} = \left[i\frac{\lambda+1+e^{-2\pi\varepsilon}}{\lambda+2}, \quad \frac{e^{-2\pi\varepsilon}-\lambda-1}{\lambda+2}, \quad -i, \quad 1\right]^T \eta.$$

The eigenvectors $\mathbf{A}_\lambda = [\mathbf{a}^{(1)} \quad \mathbf{a}^{(2)}]^T$ associated with integer eigenvalues $\lambda = n$ are

$$\mathbf{a}^{(j)} = \left[-i\frac{\lambda}{\lambda+2}, \quad -1, \quad i, \quad 1\right]^T [1+(-1)^{(j)}\alpha]\xi \quad j = 1,2$$

where η and ξ are undetermined complex constants.

From eqn (45), infinitely many eigenvalues λ and associated eigenvectors $\mathbf{A}_\lambda$ can be found. For a given eigenvalue λ, since $\lambda+1$ is also an eigenvalue (see eqns (45)), eqn (31) shows that $\mathbf{B}$ is proportional to eigenvector $\mathbf{A}_{\lambda+1}$. Therefore, $\mathbf{A}$ and $\mathbf{B}$ in eqns (31) and (32) can be expressed as

$$\mathbf{A} = k\mathbf{A}_\lambda, \quad \mathbf{B} = e\frac{k}{2a}\mathbf{A}_{\lambda+1} \tag{46}$$

where $\mathbf{A}_\lambda$, $\mathbf{A}_{\lambda+1}$ are properly normalized eigenvectors. k and e are two arbitrary constants. Substituting eqn (46) into (32), and letting $\lambda = \lambda_1 = -\frac{1}{2}+i\varepsilon$ lead to the solution for $d\mathbf{B}/d\lambda$ in the form

$$\frac{d\mathbf{B}}{d\lambda} = \frac{k}{2a}\left(\frac{d\mathbf{B}}{d\lambda}\right)_p + c\mathbf{A}_{\lambda+1} \tag{47}$$

where c is an arbitrary constant, $(d\mathbf{B}/d\lambda)_p$ is a particular solution of the following linear equations

$$\mathbf{K}_o(\lambda+1)\frac{d\mathbf{B}}{d\lambda} = \mathbf{K}_1(\lambda)\mathbf{A}_\lambda - e\frac{\partial}{\partial\lambda}[\mathbf{K}_o(\lambda+1)]\mathbf{A}_{\lambda+1}. \tag{48}$$

Note that $\|\mathbf{K}_o(\lambda+1)\| = 0$ and rank of $\mathbf{K}_o(\lambda+1)$ is seven. e can be determined for the solvability condition of the nonhomogeneous eqns (48). For a general system of algebraic equations, $\mathbf{Ax} = \mathbf{b}$, the solvability condition for the case of $\|\mathbf{A}\| = 0$ can be written as (Nayfeh, 1981, and Hilderbrand, 1965)

$$\bar{\mathbf{u}}^T\mathbf{b} = 0 \tag{49}$$

where the overbar denotes the complex conjugate, the superscript represents the transpose, and $\mathbf{u}$ is defined by

$$\mathbf{A}^*\mathbf{u} = 0 \tag{50}$$

where $\mathbf{A}^* = \bar{\mathbf{A}}^{\mathrm{T}}$.
Imposing the solvability condition on eqn (48), we have

$$e = \frac{\bar{\mathbf{u}}^{\mathrm{T}}\mathbf{K}_1(\lambda)\mathbf{A}_\lambda}{\bar{\mathbf{u}}^{\mathrm{T}}\left[\frac{\partial}{\partial\lambda}\mathbf{K}_o(\lambda+1)\right]\mathbf{A}_{\lambda+1}}. \tag{51}$$

Similarly, the higher-order terms with order $O(r^{\lambda+2})$, ... in eqn (28) for $\lambda = \lambda_1$ can be determined. Following the same procedure from eqn (47) to eqn (51), the solutions $\mathbf{dB}/\mathrm{d}\lambda$ for $\lambda = \lambda_2, \lambda_3, \ldots$ can be obtained. By adding the eigenfunctions given by eqn (29) for $\lambda = \lambda_1, \lambda_2, \ldots$, the asymptotic solution for stresses is

$$\begin{aligned}\boldsymbol{\sigma}^{(j)} = k_1\Bigg\{&r^{\lambda_1}\mathbf{S}(\theta,\lambda_1)\mathbf{a}_1^{(j)} + \frac{1}{2a}r^{\lambda_1+1}\left[e(\ln r)\mathbf{S}(\theta,\lambda_1+1)\mathbf{a}_3^{(j)} + e\left.\frac{\partial\mathbf{S}(\theta,\lambda+1)}{\partial\lambda}\right|_{\lambda=\lambda_1}\mathbf{a}_3^{(j)}\right.\\ &\left.+\mathbf{S}(\theta,\lambda+1)\left.\frac{\mathbf{db}^{(j)}}{\mathrm{d}\lambda}\right|_{\lambda=\lambda_1}\right] + r^{\lambda_1+2}[\cdots]\Bigg\}\\ &+k_2\{r^{\lambda_2}\mathbf{S}(\theta,\lambda_2)\mathbf{a}_2^{(j)} + r^{\lambda_2+1}[\cdots]\}\\ &+k_3\{r^{\lambda_3}\mathbf{S}(\theta,\lambda_3)\mathbf{a}_3^{(j)} + \cdots\} + \cdots\end{aligned} \tag{52}$$

where k_1, k_2, k_3 ... are arbitrary constants. It is understood that the real part of the right hand side of eqn (52) should be taken. Rearranging the above expansion according to the power of r, knowing that $\lambda_1 = -\frac{1}{2}+i\varepsilon$, $\lambda_2 = 0$, $\lambda_3 = \frac{1}{2}+i\varepsilon, \ldots$, we obtain, as $r \to 0$,

$$\begin{aligned}\boldsymbol{\sigma}^{(j)} = k_1 r^{\lambda_1}\mathbf{S}(\theta,\lambda_1)\mathbf{a}_1^{(j)} + k_2 r^{\lambda_2}\mathbf{S}(\theta,\lambda_2)\mathbf{a}_2^{(j)} + \frac{ek_1}{2a}r^{\lambda_3}(\ln r)\mathbf{S}(\theta,\lambda_3)\mathbf{a}_3^{(j)}\\ + r^{\lambda_3}\left[\frac{ek_1}{2a}\left.\frac{\partial\mathbf{S}(\theta,\lambda+1)}{\partial\lambda}\right|_{\lambda=\lambda_1} + \mathbf{S}(\theta,\lambda_3)\left.\frac{\mathbf{db}^{(j)}}{\mathbf{d}\lambda}\right|_{\lambda=\lambda_1} + k_3\mathbf{a}_3^{(j)}\right] + \cdots.\end{aligned}$$

It can be easily seen that r^{λ_1} and r^{λ_2} are the first- and second-order terms, respectively. $r^{\lambda_3}(\ln r)$ is the new third-term associated with the curvature effect, while r^{λ_3} becomes the fourth-order term.

INTERFACIAL CURVED CRACK UNDER ANTIPLANE SHEAR

Assume that a stress function for antiplane shear is written as

$$\Psi = r^{\lambda+1}f(\theta,\lambda) = r^{\lambda+1}\mathbf{p}(\theta,\lambda)\mathbf{a} \tag{53}$$

where $\mathbf{p} = [-\sin(\lambda+1)\theta, \quad -\cos(\lambda+1)\theta]$, $\mathbf{a} = [a_1, \quad a_2]^{\mathrm{T}}$.
Then the stresses and displacement are

$$\boldsymbol{\tau} = r^{\lambda}\mathbf{S}(\theta,\lambda)\mathbf{a}, \quad w = r^{\lambda+1}\mathbf{W}(\theta,\lambda)\mathbf{a} \tag{54}$$

where $\boldsymbol{\tau} = [\tau_r, \quad \tau_\theta]^{\mathrm{T}}$

$$\mathbf{S}(\theta,\lambda) = (\lambda+1)\begin{bmatrix} -\cos(\lambda+1)\theta & \sin(\lambda+1)\theta \\ \sin(\lambda+1)\theta & \cos(\lambda+1)\theta \end{bmatrix},$$

$$\mathbf{W}(\theta,\lambda) = -\frac{1}{\mu}[\cos(\lambda+1)\theta \quad -\sin(\lambda+1)\theta] \tag{55}$$

and μ is the shear modulus.

For an interfacial crack lying on a straight line, the boundary conditions and interfacial continuity condition are

$$\begin{aligned} &\tau_\theta^{(1)} = 0 \quad \text{at } \theta = \pi && \tau_\theta^{(1)} = \tau_\theta^{(2)} \quad \text{at } \theta = 0 \\ &\tau_\theta^{(2)} = 0 \quad \text{at } \theta = -\pi && w^{(1)} = w^{(2)} \quad \text{at } \theta = 0 \end{aligned} \tag{56}$$

or using a matrix form

$$\mathbf{K}_o(\lambda)\mathbf{A} = 0 \tag{57}$$

where

$$\mathbf{K}_o(\lambda) = \begin{bmatrix} \mathbf{N}_o\mathbf{S}(\pi,\lambda) & 0 \\ 0 & \mathbf{N}_o\mathbf{S}(-\pi,\lambda) \\ \mathbf{N}_o\mathbf{S}(0,\lambda) & -\mathbf{N}_o\mathbf{S}(0,\lambda) \\ \mathbf{W}^{(1)}(0,\lambda) & -\mathbf{W}^{(2)}(0,\lambda) \end{bmatrix}$$

$$\mathbf{N}_o = [0 \quad 1] \quad \mathbf{A} = [\mathbf{a}^{(1)} \quad \mathbf{a}^{(2)}]^{\mathrm{T}}. \tag{58}$$

Equation (57) yields the following characteristic equation and admissible eigenvalues

$$\|\mathbf{K}_o(\lambda)\| = -(\lambda+1)^3 \sin\lambda\pi \cos\lambda\pi\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}\right) = 0 \tag{59}$$

or

$$\lambda = n-\tfrac{1}{2}, \quad \lambda = n, \quad n = 0,1,2,3,\ldots$$

(a) eigenvalues $\lambda = n-1/2$ ($n = 0,1,2,3,\ldots$). The eigenvectors and stress fields are

$$\mathbf{A}_\lambda = [0 \quad 1 \quad 0 \quad 1]^{\mathrm{T}}$$

$$\tau_r^{(j)} = k_\lambda(\lambda+1)\sin(\lambda+1)\theta \quad \tau_\theta^{(j)} = k_\lambda(\lambda+1)\cos(\lambda+1)\theta$$

(b) eigenvalues $\lambda = n$ ($n = 0,1,2,3,\ldots$). The eigenvectors and stress fields are

$$\mathbf{A}_\lambda = [\mu_1 \quad 0 \quad \mu_2 \quad 0]^{\mathrm{T}}$$

$$\tau_r^{(j)} = -k_\lambda(\lambda+1)\mu_j\cos(\lambda+1)\theta \quad \tau_\theta^{(j)} = k_\lambda(\lambda+1)\mu_j\sin(\lambda+1)\theta \quad j = 1,2.$$

The asymptotic expansion of the near crack-tip stress field for each region in the case of straight crack is

$$\boldsymbol{\tau}^{(j)} = k_{-1/2}r^{-1/2}\mathbf{S}(\theta,-1/2)\mathbf{a}' + k_0\mathbf{S}(\theta,0)\mathbf{a}'' + k_{1/2}r^{1/2}\mathbf{S}(\theta,1/2)\mathbf{a}' + O(r) \tag{60}$$

where

$$\mathbf{a}^I = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \mathbf{a}^{II} = \begin{bmatrix} \mu_j \\ 0 \end{bmatrix}. \tag{61}$$

For an interface crack lying on a smooth curved line, the boundary conditions and continuity condition on the interface are

$$\begin{aligned} &\mathbf{t}^{(1)} = 0 \quad \text{on } \Gamma_1 \qquad \mathbf{t}^{(1)} = -\mathbf{t}^{(2)} \quad \text{on } \Gamma_3 \\ &\mathbf{t}^{(2)} = 0 \quad \text{on } \Gamma_2 \qquad w^{(1)} = w^{(2)} \quad \text{on } \Gamma_3 \end{aligned} \tag{62}$$

where $\mathbf{t}$ is a surface traction vector

$$\mathbf{t} = \mathbf{N}\boldsymbol{\tau} \quad \mathbf{N} = [n_r, \quad n_\theta] \tag{63}$$

(n_r, n_θ) are polar components of the unit normal $\mathbf{n}$ to the boundaries. For an interfacial crack lying on a circular arc, the eqns (23–25) hold. Using (25) leads to

$$\mathbf{N} = n_\theta\{\mathbf{N}_o - \mathbf{N}_1(\theta_1 r + 2\theta_2 r^2 + \cdots)\} \tag{64}$$

where

$$\mathbf{N}_o = [0 \quad 1] \quad \mathbf{N}_1 = [1 \quad 0].$$

On the curved boundaries, expanding $\mathbf{S}$ and $\mathbf{W}$ of eqn (54) in a Taylor series form in r we obtain

$$\begin{aligned} \mathbf{S}(\theta,\lambda) &= \mathbf{S}(\theta_o,\lambda) + \theta_1\mathbf{S}'(\theta_o,\lambda)r + [2\theta_2\mathbf{S}'(\theta_o,\lambda) + \theta_1^2\mathbf{S}''(\theta_o,\lambda)]r^2/2 + \cdots \\ \mathbf{W}(\theta,\lambda) &= \mathbf{W}(\theta_o,\lambda) + \theta_1\mathbf{W}'(\theta_o,\lambda)r + [2\theta_2\mathbf{W}'(\theta_o,\lambda) + \theta_1^2\mathbf{W}''(\theta_o,\lambda)]r^2/2 + \cdots. \end{aligned} \tag{65}$$

To satisfy the boundary conditions and interface conditions, consider the following modified stress function for a given λ

$$\boldsymbol{\Psi} = r^{\lambda+1}\mathbf{p}(\theta,\lambda)\mathbf{a} + r^{\lambda+2}\mathbf{p}(\theta,\lambda+1)\mathbf{b} + r^{\lambda+3}\mathbf{p}(\theta,\lambda+2)\mathbf{c} + \cdots. \tag{66}$$

Then the expressions for stress and displacement can be written as

$$\boldsymbol{\tau} = r^{\lambda}\mathbf{S}(\theta,\lambda)\mathbf{a} + r^{\lambda+1}\mathbf{S}(\theta,\lambda+1)\mathbf{b} + r^{\lambda+2}\mathbf{S}(\theta,\lambda+2)\mathbf{c} + \cdots \tag{67}$$

$$w = r^{\lambda+1}\mathbf{W}(\theta,\lambda)\mathbf{a} + r^{\lambda+2}\mathbf{W}(\theta,\lambda+1)\mathbf{b} + r^{\lambda+3}\mathbf{W}(\theta,\lambda+2)\mathbf{c} + \cdots. \tag{68}$$

Substituting (67), (68) into (62), using (64)–(65), collecting the terms with the same power of r and equating the coefficient of each power of r in each equation to zero, we have the system of linear equations

$$\mathbf{K}_o(\lambda)\mathbf{A} = 0 \tag{69}$$

$$\mathbf{K}_o(\lambda+1)\mathbf{B} = \mathbf{K}_1(\lambda)\mathbf{A} \tag{70}$$

where $\mathbf{B} = [\mathbf{b}^{(1)}, \quad \mathbf{b}^{(2)}]^{\mathrm{T}}$, $\mathbf{K}_o(\lambda)$ is given in eqn (58) and

$$\mathbf{K}_1(\lambda) = \begin{bmatrix} \mathbf{N}_1\mathbf{S}(\pi,\lambda)-\mathbf{N}_o\mathbf{S}'(\pi,\lambda) & 0 \\ 0 & \mathbf{N}_1\mathbf{S}(-\pi,\lambda)-\mathbf{N}_o\mathbf{S}'(-\pi,\lambda) \\ -[\mathbf{N}_1\mathbf{S}(0,\lambda)-\mathbf{N}_o\mathbf{S}'(0,\lambda)] & \mathbf{N}_1\mathbf{S}(0,\lambda)-\mathbf{N}_o\mathbf{S}'(0,\lambda) \\ \mathbf{W}^{(1)}(0,\lambda) & -\mathbf{W}^{(2)}(0,\lambda) \end{bmatrix}. \quad (71)$$

Comparing eqn (69) with eqn (57), it is clearly seen that λ and $\mathbf{A}$ are identical to those of the crack with straight boundaries; while the vector $\mathbf{B}$ is determined in terms of $\mathbf{A}$ from eqn (70) for each λ. Using the solvability condition of nonhomogeneous eqn (70) and solving eqn (70) for $\lambda = -1/2$, we obtain $\mathbf{B}_{\lambda=1/2}$. Substituting $\mathbf{A}$ and $\mathbf{B}$ back into (67) with $\lambda = -1/2$ and adding the second term with order r°, the first three-term asymptotic expansion of the near circular crack-tip stress field in the following form for each material is:

$$\boldsymbol{\tau}^{(j)} = k_{-1/2}r^{-1/2}\mathbf{S}(\theta,-1/2)\mathbf{a}'+k_0\mathbf{S}(\theta,0)\mathbf{a}''+r^{1/2}\mathbf{S}(\theta,1/2)\left[k_{1/2}\mathbf{a}'+\frac{k_{-1/2}}{2a}\mathbf{b}\right] \quad (72)$$

where

$$\mathbf{b} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}.$$

It is worth noting that if the curvature changes its sign (for instance, from convex to concave), the vector $\mathbf{b}$ also alters its sign. In order to validate the asymptotic solution of curve crack-tip stress field, a circular-arc crack lying along the interface in an infinite body, which is subjected to the externally applied antiplane stress, $\tau_x = \tau_x^\infty$, $\tau_y = \tau_y^\infty$ is considered. The complex exact solution for the stress field has been obtained by Yang and Yuan (1995) and is expressed as:

$$\tau_\rho^{(j)} - i\tau_\varphi^{(j)} = \frac{\tau^\infty a}{\mu_1+\mu_2}\frac{1}{r}\mathrm{Re}\left[\mu_j\left(-\frac{a}{z}e^{i\beta}+\frac{z}{a}e^{-i\beta}\right)-\mu_1 a\frac{\left(\frac{a}{z}-\cos\alpha\right)e^{i\beta}+\left(\cos\alpha-\frac{z}{a}\right)\frac{z}{a}e^{-i\beta}}{(z-a\,e^{i\alpha})^{1/2}(z-a\,e^{-i\alpha})^{1/2}}\right] \quad (73)$$

where $\tau_y^\infty = \tau^\infty \sin\beta$, $\tau_x^\infty = \tau^\infty\cos\beta$. α and β in this subsection denote the half of the crack angle and the loading direction with respect to the x-axis, respectively. (ρ,φ) are the polar coordinates with origin at the circular center and $z = \rho\,e^{i\varphi}$.

Expanding (73) for small r from the crack tip $z = a\,e^{i\alpha}$ and introducing a local polar coordinate system (r,θ) together with stress transformation, we have the three-term expansion of the stress field

$$\begin{Bmatrix} \tau_r^{(j)} \\ \tau_\theta^{(j)} \end{Bmatrix} = k_{-1/2}r^{-1/2}\mathbf{S}(\theta,-1/2)\mathbf{a}'+k_0\mathbf{S}(\theta,0)\mathbf{a}''+k_{1/2}r^{1/2}\mathbf{S}(\theta,1/2)\begin{Bmatrix} \frac{k_{-1/2}}{4ak_{1/2}} \\ 1 \end{Bmatrix} \quad (74)$$

or

$$\tau_r^{(j)} = k_{-1/2}r^{-1/2}\tilde{\tau}_r^{(1)}+k_0\mu_\alpha\tilde{\tau}_r^{(2)}+k_{1/2}r^{1/2}\tilde{\tau}_r^{(3)}$$
$$\tau_\theta^{(j)} = k_{-1/2}r^{-1/2}\tilde{\tau}_\theta^{(1)}+k_0\mu_\alpha\tilde{\tau}_\theta^{(2)}+k_{1/2}r^{1/2}\tilde{\tau}_\theta^{(3)} \quad (75)$$

where

$$k_{-1/2} = -2\frac{\mu_1}{\mu_1+\mu_2}\sqrt{2a\sin\alpha}\left(\tau_x^\infty\cos\frac{\alpha}{2}+\tau_y^\infty\sin\frac{\alpha}{2}\right) = -\frac{2\mu_1\tau_0}{\mu_1+\mu_2}\sqrt{2a\sin\alpha}\cos\left(\beta-\frac{\alpha}{2}\right)$$

$$k_0 = \frac{-2}{\mu_1+\mu_2}(-\tau_x^\infty\sin\alpha+\tau_y^\infty\cos\alpha) = -\frac{2\tau_0}{\mu_1+\mu_2}\sin(\beta-\alpha)$$

$$k_{1/2} = \frac{-\mu_1}{\mu_1+\mu_2}\sqrt{\frac{1}{2a\sin\alpha}}\left[\tau_x^\infty(3\cos\alpha-2)\cos\frac{\alpha}{2}+\tau_y^\infty(3\cos\alpha+2)\sin\frac{\alpha}{2}\right]$$

$$= -\frac{\mu_1\tau_0}{\mu_1+\mu_2}\sqrt{\frac{1}{2a\sin\alpha}}\left[\cos\alpha\cos\left(\beta-\frac{\alpha}{2}\right)+2\sin\alpha\sin\left(\beta-\frac{\alpha}{2}\right)\right] \tag{76}$$

$$\tilde{\tau}_r^{(1)} = \frac{1}{2}\sin\frac{\theta}{2} \quad \tilde{\tau}_r^{(2)} = -\cos\theta \quad \tilde{\tau}_r^{(3)} = \frac{3}{2}\left(\sin\frac{3}{2}\theta+\frac{k_{-1/2}}{4ak_{1/2}}\cos\frac{3}{2}\theta\right)$$

$$\tilde{\tau}_\theta^{(1)} = \frac{1}{2}\cos\frac{\theta}{2} \quad \tilde{\tau}_\theta^{(2)} = \sin\theta \quad \tilde{\tau}_\theta^{(3)} = \frac{3}{2}\left(\cos\frac{3}{2}\theta-\frac{k_{-1/2}}{4ak_{1/2}}\sin\frac{3}{2}\theta\right) \tag{77}$$

$$\frac{k_{-1/2}}{4ak_{1/2}} = \frac{\sin\alpha\cos\left(\beta-\frac{\alpha}{2}\right)}{\cos\alpha\cos\left(\beta-\frac{\alpha}{2}\right)+2\sin\alpha\sin\left(\beta-\frac{\alpha}{2}\right)}. \tag{78}$$

It is readily seen that eqns (74) and (75) coincide with eqn (72). It follows from (77) that the angular distributions for the first two terms, $\tilde{\tau}_r^{(1)}$, $\tilde{\tau}_\theta^{(1)}$, $\tilde{\tau}_r^{(2)}$, $\tilde{\tau}_\theta^{(2)}$, are identical to those for straight cracks, while the angular distributions for the third term, $\tilde{\tau}_r^{(3)}$, $\tilde{\tau}_\theta^{(3)}$, are different. Figure 4 shows the angular distribution of stress components for different values of $\xi = k_{-1/2}/(2ak_{1/2})$ with $\beta = \alpha/2$. Comparing with the distribution of the straight crack, the presence of the curvature has significant effect on the character of the angular distribution.

When $a \to \infty$, the curved crack approaches the case of straight cracks. From eqn (76), we obtain

$$k^s_{-1/2} = \lim_{a\to\infty} k_{-1/2} = \frac{2\mu_2\tau_x^\infty}{\mu_1+\mu_2}\sqrt{2a\alpha}, \quad k^s_0 = \lim_{a\to\infty} k_0 = \frac{2\tau_y^\infty}{\mu_1+\mu_2}$$

$$k^s_{1/2} = \lim_{a\to\infty} k_{1/2} = \frac{2\mu_2\tau_x^\infty}{\mu_1+\mu_2}\sqrt{\frac{1}{2a\alpha}}, \quad \lim_{a\to\infty}\frac{k_{-1/2}}{4ak_{1/2}} = 0. \tag{79}$$

Note that $2a\alpha$ is the length of the crack in eqn (79). Comparing amplitudes k_i for the curved crack with k_i^s for the planar crack with the same crack length yields

$$\frac{k_{-1/2}}{k^s_{-1/2}} = \sqrt{\frac{\sin\alpha}{\alpha}}\frac{\cos\left(\beta-\frac{\alpha}{2}\right)}{\cos\beta}, \quad \frac{k_0}{k^s_0} = \frac{\sin(\beta+\alpha)}{\sin\beta}$$

$$\frac{k_{1/2}}{k^s_{1/2}} = \sqrt{\frac{\alpha}{\sin\alpha}}\left[3\cos\alpha\cos\left(\beta-\frac{\alpha}{2}\right)-2\cos\left(\beta+\frac{\alpha}{2}\right)\right]. \tag{80}$$

CONCLUSIONS

The role of curvature on the stress distribution of curved interfacial crack between dissimilar isotropic solids has been studied. Using asymptotic expansions for the stress and

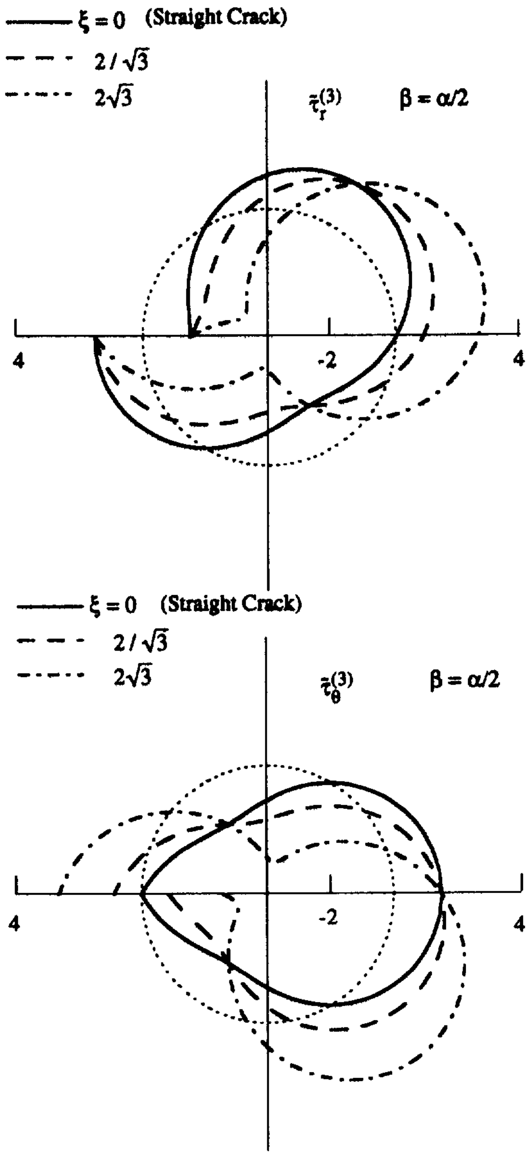

Fig. 4. Stress angular distribution of the third term asymptotic expansion for circular interfacial crack under antiplane shear.

curved geometry, the detailed stress and displacement distribution near the crack tip can be investigated. Based on the analysis from the composite under in-plane and antiplane shear loading, the following conclusions may be drawn:

(1) The stress exponents and associated stress angular distributions for the first two term asymptotic expansions are not affected by the constant radius of curvature along the crack interface.
(2) The curvature effect enters in the third-term asymptotic expansion of the stress distribution for circular interfacial crack.
(3) Under in-plane loading, the third-term asymptotic solution may contain the $r^{1/2}(\ln r)$ function. For the interfacial crack with closed tips, the $r^{1/2}(\ln r)$ term exists if the second Dundurs' parameter β is not zero.
(4) It can be deduced based on the approach that, for any crack along a smooth curve, the singular field excluding the amplitude is identical to that of a straight crack along the tangent of the crack. For other types of curved geometry, the curvature may contribute to the second-term asymptotic solution.

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APPENDIX

$$\mathbf{K}_o(\lambda) = \begin{bmatrix} \Sigma_{r\theta}(\pi,\lambda) & 0 \\ 0 & \Sigma_{r\theta}(-\pi,\lambda) \\ \Sigma_{\theta}(\pi,\lambda) & -\Sigma_{\theta}(-\pi,\lambda) \\ U_{\theta}^{(1)}(\pi,\lambda) & -U_{\theta}^{(2)}(-\pi,\lambda) \\ \Sigma_{r\theta}(0,\lambda) & -\Sigma_{r\theta}(0,\lambda) \\ \Sigma_{\theta}(0,\lambda) & -\Sigma_{\theta}(0,\lambda) \\ U_{r}^{(1)}(0,\lambda) & -U_{r}^{(2)}(0,\lambda) \\ U_{\theta}^{(1)}(0,\lambda) & -U_{\theta}^{(2)}(0,\lambda) \end{bmatrix}$$

$$
= \begin{bmatrix}
(\lambda+1)(\lambda+2)c & -(\lambda+1)(\lambda+2)s & \lambda(\lambda+1)c & -\lambda(\lambda+1)s & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & (\lambda+1)(\lambda+2)c & (\lambda+1)(\lambda+2)s & \lambda(\lambda+1)c & \lambda(\lambda+1)s \\
-(\lambda+1)(\lambda+2)s & -(\lambda+1)(\lambda+2)c & -(\lambda+1)(\lambda+2)s & -(\lambda+1)(\lambda+2)c & -(\lambda+1)(\lambda+2)s & (\lambda+1)(\lambda+2)c & -(\lambda+1)(\lambda+2)s & (\lambda+1)(\lambda+2)c \\
\frac{\lambda+2}{2\mu_1}c & -\frac{\lambda+2}{2\mu_1}s & \frac{\kappa_1+\lambda+1}{2\mu_1}c & -\frac{\kappa_1+\lambda+1}{2\mu_1}s & -\frac{\lambda+2}{2\mu_2}c & -\frac{\lambda+2}{2\mu_2}s & -\frac{\kappa_2+\lambda+1}{2\mu_2}c & -\frac{\kappa_2+\lambda+1}{2\mu_2}s \\
-(\lambda+1)(\lambda+2) & 0 & -\lambda(\lambda+1) & 0 & (\lambda+1)(\lambda+2) & 0 & \lambda(\lambda+1) & 0 \\
0 & (\lambda+1)(\lambda+2) & 0 & (\lambda+1)(\lambda+2) & 0 & -(\lambda+1)(\lambda+2) & 0 & -(\lambda+1)(\lambda+2) \\
0 & -\frac{\lambda+2}{2\mu_1} & 0 & \frac{\kappa_1-\lambda-1}{2\mu_1} & 0 & \frac{\lambda+2}{2\mu_2} & 0 & -\frac{\kappa_2-\lambda-1}{2\mu_2} \\
-\frac{\lambda+2}{2\mu_1} & 0 & -\frac{\kappa_1+\lambda+1}{2\mu_1} & 0 & \frac{\lambda+2}{2\mu_2} & 0 & \frac{\kappa_2+\lambda+1}{2\mu_2} & 0
\end{bmatrix}
$$

$c = \cos\lambda\pi$, $s = \sin\lambda\pi$.

$$
\mathbf{K}_1(\lambda) = \begin{bmatrix}
\Sigma'_{r\theta}(\pi,\lambda)+\Sigma_\theta(\pi,\lambda)-\Sigma_r(\pi,\lambda) & 0 \\
0 & \Sigma'_{r\theta}(-\pi,\lambda)+\Sigma_\theta(-\pi,\lambda)-\Sigma_r(-\pi,\lambda) \\
\Sigma'_\theta(\pi,\lambda)-2\Sigma_{r\theta}(\pi,\lambda) & -[\Sigma'_\theta(-\pi,\lambda)-2\Sigma_{r\theta}(-\pi,\lambda)] \\
[U'_\theta(\pi,\lambda)-U_r(\pi,\lambda)]^{(1)} & -[U'_\theta(-\pi,\lambda)-U_r(-\pi,\lambda)]^{(2)} \\
-[\Sigma'_{r\theta}(0,\lambda)-\Sigma_r(0,\lambda)\} & \Sigma'_{r\theta}(0,\lambda)-\Sigma_r(0,\lambda) \\
-[\Sigma'_\theta(0,\lambda)-\Sigma_{r\theta}(0,\lambda)] & \Sigma'_\theta(0,\lambda)-\Sigma_{r\theta}(0,\lambda) \\
-[U'_r(0,\lambda)]^{(1)} & [U'_r(0,\lambda)]^{(2)} \\
-[U'_\theta(0,\lambda)]^{(1)} & [U'_\theta(0,\lambda)]^{(2)}
\end{bmatrix}
$$

$$
K_1(\lambda)|_{\lambda=-1/2}\mathbf{A}_{-1/2} = \begin{bmatrix}
-(6+\beta) \\
(6-\beta) \\
0 \\
-\frac{1}{2\mu_1}[7+3\kappa_1-3\beta(\kappa_1-1)]-\frac{1}{2\mu_2}[7+3\kappa_2+3\beta(\kappa_2-1)] \\
0 \\
0 \\
\frac{1}{2\mu_1}[\kappa_1-1+\beta(1+\kappa_1)]-\frac{1}{2\mu_2}[5-\kappa_2-\beta(1+\kappa_2)] \\
0
\end{bmatrix}
$$

$$\frac{\partial}{\partial \lambda}[\mathbf{K}_o(\lambda+1)]|_{\lambda=-1/2}\mathbf{A}_{1/2} = \begin{bmatrix} 2\pi \\ 2\pi \\ 0 \\ \dfrac{\pi}{3\mu_1}[1-\kappa_1+\beta(1+\kappa_1)] - \dfrac{\pi}{3\mu_2}[1-\kappa_2+\beta(1+\kappa_2)] \\ -\dfrac{8}{5}\beta \\ 0 \\ 0 \\ -\dfrac{4}{15}\beta\left(\dfrac{1}{\mu_1}+\dfrac{1}{\mu_2}\right) \end{bmatrix}.$$